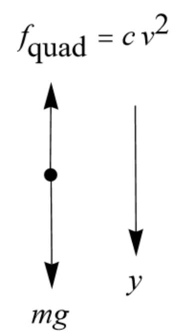
Problem 2.35

(a) Fill in the details of the arguments leading from the equation of motion (2.52) to Equations (2.57) and (2.58) for the velocity and position of a dropped object subject to quadratic air resistance. Be sure to do the two integrals involved. (The results of Problem 2.34 will help.)
(b) Tidy the two equations by introducing the parameter $\tau = v_{\text{ter}}/g$. Show that when $t = \tau$, v has reached 76% of its terminal value. What are the corresponding percentages when $t = 2\tau$ and 3τ? **(c)** Show that when $t \gg \tau$, the position is approximately $y \approx v_{\text{ter}}t + \text{const}$. [*Hint:* The definition of $\cosh x$ (Problem 2.33) gives you a simple approximation when x is large.] **(d)** Show that for t small, Equation (2.58) for the position gives $y \approx \frac{1}{2}gt^2$. [Use the Taylor series for $\cosh x$ and for $\ln(1+\delta)$.]

Solution

Part (a)

Draw a free-body diagram for a mass falling down in a medium with quadratic air resistance. Let the positive y-direction point downward.



Apply Newton's second law in the y-direction.

$$\sum F_y = ma_y$$

Let $v_y = v$ to simplify the notation.

$$mg - cv^2 = m\frac{dv}{dt} \tag{2.52}$$

This is Equation (2.52) on page 60. The terminal speed occurs when the velocity reaches equilibrium.

$$mg - cv_{\text{ter}}^2 = m(0)$$

Solve for v_{ter}, the terminal velocity.

$$v_{\text{ter}} = \sqrt{\frac{mg}{c}}$$

To get v, solve Equation (2.52) by separating variables.

$$c\left(\frac{mg}{c} - v^2\right) = m\frac{dv}{dt}$$

$$\frac{c}{m}\,dt = \frac{dv}{\frac{mg}{c} - v^2}$$

Integrate both sides definitely, assuming that at $t = 0$ the velocity is zero.

$$\int_0^t \frac{c}{m}\,dt' = \int_0^v \frac{dv'}{\frac{mg}{c} - v'^2} \tag{1}$$

Make the following substitution in the integral on the right side.

$$v' = \sqrt{\frac{mg}{c}}\tanh\theta$$

$$dv' = \sqrt{\frac{mg}{c}}\operatorname{sech}^2\theta\,d\theta$$

Consequently, equation (1) becomes

$$\frac{c}{m}(t-0) = \int_{\tanh^{-1}\left(\frac{0}{\sqrt{\frac{mg}{c}}}\right)}^{\tanh^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}}}\right)} \frac{\sqrt{\frac{mg}{c}}\operatorname{sech}^2\theta\,d\theta}{\frac{mg}{c}(1-\tanh^2\theta)}$$

$$\frac{c}{m}t = \frac{1}{\sqrt{\frac{mg}{c}}}\int_0^{\tanh^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}}}\right)} \frac{\operatorname{sech}^2\theta\,d\theta}{\operatorname{sech}^2\theta}$$

$$= \sqrt{\frac{c}{mg}}\int_0^{\tanh^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}}}\right)} d\theta$$

$$= \sqrt{\frac{c}{mg}}\left[\tanh^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}}}\right) - 0\right]$$

$$= \sqrt{\frac{c}{mg}}\tanh^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}}}\right).$$

Solve for v.

$$\sqrt{\frac{cg}{m}}\,t = \tanh^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}}}\right)$$

$$\tanh\left(\sqrt{\frac{cg}{m}}\,t\right) = \frac{v}{\sqrt{\frac{mg}{c}}}$$

Therefore, the velocity in a medium with quadratic air resistance is

$$\boxed{v(t) = \sqrt{\frac{mg}{c}}\tanh\left(\sqrt{\frac{cg}{m}}\,t\right).}$$

To get an equation involving the position, replace v with dy/dt.

$$\frac{dy}{dt} = \sqrt{\frac{mg}{c}} \tanh\left(\sqrt{\frac{cg}{m}}\, t\right)$$

Separate variables to solve for y.

$$dy = \sqrt{\frac{mg}{c}} \tanh\left(\sqrt{\frac{cg}{m}}\, t\right) dt$$

Integrate both sides definitely, assuming that at $t = 0$ the position is zero.

$$\int_0^y dy' = \int_0^t \sqrt{\frac{mg}{c}} \tanh\left(\sqrt{\frac{cg}{m}}\, t'\right) dt'$$

$$y - 0 = \sqrt{\frac{mg}{c}} \int_0^t \tanh\left(\sqrt{\frac{cg}{m}}\, t'\right) dt' \tag{2}$$

Make the following substitution in the remaining integral.

$$u = \sqrt{\frac{cg}{m}}\, t'$$

$$du = \sqrt{\frac{cg}{m}}\, dt' \quad \rightarrow \quad dt' = \sqrt{\frac{m}{cg}}\, du$$

Consequently, equation (2) becomes

$$\begin{aligned} y(t) &= \sqrt{\frac{mg}{c}} \int_0^{\sqrt{\frac{cg}{m}}\, t} (\tanh u) \left(\sqrt{\frac{m}{cg}}\, du\right) \\ &= \frac{m}{c} \int_0^{\sqrt{\frac{cg}{m}}\, t} \tanh u \, du \\ &= \frac{m}{c} \ln \cosh u \Big|_0^{\sqrt{\frac{cg}{m}}\, t} \\ &= \frac{m}{c} \ln \frac{\cosh \sqrt{\frac{cg}{m}}\, t}{\cosh 0}. \end{aligned}$$

Therefore, the position in a medium with quadratic air resistance is

$$\boxed{y(t) = \frac{m}{c} \ln \left[\cosh\left(\sqrt{\frac{cg}{m}}\, t\right)\right].}$$

Part (b)

In terms of $v_{\text{ter}} = \sqrt{mg/c}$ and $\tau = v_{\text{ter}}/g$, the velocity is

$$v(t) = v_{\text{ter}} \tanh\left(\frac{g}{v_{\text{ter}}}t\right) = v_{\text{ter}} \tanh\left(\frac{t}{\tau}\right), \tag{2.57}$$

and the position is

$$y(t) = \frac{v_{\text{ter}}^2}{g} \ln\left[\cosh\left(\frac{g}{v_{\text{ter}}}t\right)\right] = v_{\text{ter}}\tau \ln\left[\cosh\left(\frac{t}{\tau}\right)\right]. \tag{2.58}$$

These are Equations (2.57) and (2.58) on page 61. If $t = \tau$, then the velocity is about 76% of the terminal velocity.

$$v(\tau) = v_{\text{ter}} \tanh 1 \approx 0.76 v_{\text{ter}}$$

If $t = 2\tau$, then the velocity is about 96% of the terminal velocity.

$$v(2\tau) = v_{\text{ter}} \tanh 2 \approx 0.96 v_{\text{ter}}$$

If $t = 3\tau$, then the velocity is about 99.5% of the terminal velocity.

$$v(3\tau) = v_{\text{ter}} \tanh 3 \approx 0.995 v_{\text{ter}}$$

Part (c)

If $t \gg \tau$, then

$$\begin{aligned}
y(t) &= v_{\text{ter}}\tau \ln\left[\cosh\left(\frac{t}{\tau}\right)\right] \\
&= v_{\text{ter}}\tau \ln\left[\frac{\exp\left(\frac{t}{\tau}\right) + \cancelto{0}{\exp\left(-\frac{t}{\tau}\right)}}{2}\right] \\
&\approx v_{\text{ter}}\tau \ln\left[\frac{1}{2}\exp\left(\frac{t}{\tau}\right)\right] \\
&\approx v_{\text{ter}}\tau \left[\ln\frac{1}{2} + \ln\exp\left(\frac{t}{\tau}\right)\right] \\
&\approx v_{\text{ter}}\tau \left(-\ln 2 + \frac{t}{\tau}\right) \\
&\approx \underbrace{-v_{\text{ter}}\tau \ln 2}_{\text{const}} + v_{\text{ter}}t.
\end{aligned}$$

Part (d)

The Taylor series for hyperbolic cosine and natural logarithm about $x = 0$ are as follows.

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

If t is very small ($t \ll 1$), then the hyperbolic cosine function can be reasonably approximated by the first few terms of its Taylor series about zero. Since the goal is to get $y \approx \frac{1}{2}gt^2$, only keep the first two terms of the series. The higher-order terms are negligible compared to the first two.

$$\begin{aligned} y(t) &= v_{\text{ter}}\tau \ln\left[\cosh\left(\frac{t}{\tau}\right)\right] \\ &\approx v_{\text{ter}}\tau \ln\left[1 + \frac{1}{2!}\left(\frac{t}{\tau}\right)^2\right] \\ &\approx v_{\text{ter}}\tau \ln\left(1 + \frac{t^2}{2\tau^2}\right) \end{aligned}$$

Since t is very small, $t^2/(2\tau^2)$ is close to zero, so the natural logarithm can be reasonably approximated by the first few terms of its Taylor series about zero. Since the goal is to get $y \approx \frac{1}{2}gt^2$, only keep the first term of the series. The higher-order terms are negligible compared to the first.

$$\begin{aligned} y(t) &\approx v_{\text{ter}}\tau\left(\frac{t^2}{2\tau^2}\right) \\ &\approx \frac{1}{2}\left(\frac{v_{\text{ter}}}{\tau}\right)t^2 \\ &\approx \frac{1}{2}gt^2 \end{aligned}$$